

Non-classicalities via perturbing local unitary operations

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Abstract. We study the nonclassical correlations in a two-qubit state by the perturbing local unitary operation method. We find that the definitions of various non-classicalities including quantum discord (QD), measurement-induced nonlocality (MIN) and so on usually do not have a unique definition when expressed as the perturbation of local unitary operations, so a given non-classicality can lead to different definitions of its dual non-classicality. In addition, it is shown that QD and MIN are not the corresponding dual expressions in a simple set of unitary operations, even though they are in their original definitions. In addition, we also consider the non-classicalities in general $2 \otimes d$ dimensional systems.

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1 Introduction

Quantum correlation is one of the most intriguing features of quantum mechanics and plays an important role in quantum information. The quantification of quantum correlation has attracted much attention in recent times. Among the many measures of quantum correlation such as entanglement [1], quantum discord [2,3], the information deficit [4], the measurement induced nonlocality [5] and so on, entanglement and quantum discord have been more extensively researched. For example, entanglement has been investigated extensively and intensively over the last two decades [1] and quantum discord seems to be attracting increasing interest. However, quantum entanglement and quantum discord are different, not only in that quantum discord can be present in separable states, but also in that it can be increased in local operations and classical communications [6]. Quantum entanglement has been identified as an important physical resource in quantum information processing tasks (QIPTs), but it has been found that some QIPTs without any entanglement could also display quantum advantages if there exists quantum discord [7,8,9,10]. This could shed new light on the role of quantum discord in quantum computing, hence it could be one of the main reasons for the recent widespread research on quantum discord in various fields such as dynamic evolution [11,12,13,14,15], Maxwell's demon [16], relativistic effects in quantum information theory [17,18], quantum phase transitions [19,20,21,22], biological systems [23] and so on.

The original definition of quantum discord is information-theoretical [24,25]. However, the analytic expression is only available for some special states [26,27,28,29,30,31,32,33]. For this reason, the geometric version of quantum dis-

cord (GD) based on distance measurements was introduced for a two-qubit system in 2010 [3]. It provided a better method for analytically evaluating $2 \otimes d$ dimensional states [34]. Recently, it has been pointed out that the Frobenius norm is unable to account for experimentally observed contractilities [6,35], namely, geometric discord will be increased under a non-unitary evolution that is described by a completely positive local operation [6,36]. In fact, there are two ways to define distances in vector spaces: 1) by considering some properly defined norm and the metrics that it induces; 2) by considering a proper metric, not directly related to a norm, such as the Fubini-Study metrics and the Bures metrics in classical cases [37]. In general, these metrics are, by construction, Riemannian and contractive, while problems can arise when one considers case 1). Indeed, in finite-dimensional vector spaces, metrics are induced starting from the Schatten p -norms, that are the finite-dimensional precursors of the general p -norms in L_p spaces. The case $p = 1$ corresponds to the trace norm (contractive but non-Riemannian); the case $p = 2$ corresponds to the Hilbert-Schmidt norm (not even contractive); the case $p = \infty$ corresponds to the *sup* norm. Only norms with $p < 2$ are contractive. Thus, all kinds of distance measurements of quantum correlations have advantages and disadvantages [38]. This way of thinking about quantum correlation has been systematically studied in Ref. [39]. Some other definitions of quantum correlation considering the contractivity have also been proposed [40,41,42]. At the same time, some works attempt to compute distance measures of quantum correlations based on the trace norm [43,44].

In addition, different definitions related to quantum correlations have also been developed to different extents. For example, quantum discord with two-side measurements has been considered [45]; the measurement-induced nonlo-

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cality (MIN) is introduced by considering the maximal distance between the original and the measured density matrices [5]. In particular, geometric quantum discord (even entanglement [46,47]) has been shown to be redefined via the perturbation of local unitary operations, which also offered an alternative understanding of quantum discord [48,49,50]. Thus it is natural to ask whether the MIN can be reconstructed by similar consideration of the perturbation under local unitary operations and whether the perturbation of local unitary operations can lead to other interesting phenomena?

In this paper we consider non-classicalities through perturbation under local unitary operations. We divide the set of unitary operations into several sets which are made up of the traceless unitary operations, the cyclic unitary operations, the special unitary operations and the general unitary operations, respectively. It is well known that GD and MIN are dual definitions with respect to local measurements [5,48,49]. However, we first find that, even though GD can be redefined as the minimal distance between the state of interest and the one that is perturbed by the local unitary operations which are in the set of cyclic unitary operations, MIN cannot be reached by the dual definition (maximization) in the same set. On the contrary, it will arrive at a new quantity which we call the generalized MIN (GMIN) in contrast to MIN. Second, the definition of GD based on the perturbation under local unitary operations are not unique. The minimization of the distance on both the traceless unitary operation set and the cyclic unitary operation set can lead to GD. Finally, we also find that there is no simple unitary operation set (but a relatively complex special set) that is directly related to the MIN and GD by the dual optimization.

This paper is organized as follows. In Sec. II, we give a brief classification of the unitary operators used in this work. In Sec. III, we give the expressions for various non-classicalities based on different sets of unitary operations. In Sec. IV, we study the connection between GD, MIN and GMIN. In Sec. V, we expand our results to the $2 \otimes d$ dimensional quantum systems and highlight our conclusions.

2 Sets of unitary operations

Let us consider the unitary operations in a 2-dimensional Hilbert space. Any unitary U can be written up to a constant phase as

$$U = n_0 I_2 + i \mathbf{n} \cdot \boldsymbol{\sigma}, \quad (1)$$

where I_n denotes the n -dimensional identity, $n_0 \in \mathbb{R}$, $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with σ_i the Pauli matrices. The unitary property of U requires

$$\sum_{k=0}^3 n_k^2 = 1. \quad (2)$$

Thus, based on the different parameters, one can divide the set of unitary matrices into different subsets. At first, we would like to use S_A to denote a general unitary matrix

given in Eq. (1). If $n_0 = 0$, one will find that U is traceless. This traceless unitary operator set is denoted by S_T . Now let us consider an arbitrary two-qubit state ρ_{AB} in the Bloch representation as [51]

$$\rho_{AB} = \frac{1}{4} [I_4 + (\mathbf{r} \cdot \boldsymbol{\sigma}_A) \otimes I_2 + I_2 \otimes (\mathbf{s} \cdot \boldsymbol{\sigma}_B) + \sum_{i,j} T_{ij} \sigma_i \otimes \sigma_j], \quad (3)$$

where \mathbf{r}, \mathbf{s} and T_{ij} are the local Bloch vectors and the correlation tensor, respectively. Then, the reduced density matrix can be given by

$$\rho_A = \text{Tr}_B \rho_{AB} = \frac{1}{2} \begin{pmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix}. \quad (4)$$

Suppose a unitary operation U that is performed on ρ_A satisfies

$$[U, \rho_A] = 0, \quad (5)$$

then such unitary operations U will compose a cyclic unitary set S_C subject to ρ_A . In addition, let us use S_S to describe the set made up of some special unitary operations which will be given in Theorem 3 in the following. Thus, based on these unitary operator sets, we will give the different formulations of the non-classicalities in the next section. Before proceeding, we would like to give a lemma that is very useful in this context.

Lemma 1. Eq. (5) is equivalent to

$$c_1 \mathbf{n} = c_2 \mathbf{r}, \quad c_i \in \mathbb{R}, \quad (6)$$

Proof. Inserting Eq. (1) and Eq. (4) into Eq. (5), one will arrive at

$$\begin{aligned} (I_2 + \mathbf{r} \cdot \boldsymbol{\sigma}) (n_0 I_2 + i \mathbf{n} \cdot \boldsymbol{\sigma}) &= (n_0 I_2 + i \mathbf{n} \cdot \boldsymbol{\sigma}) (I_2 + \mathbf{r} \cdot \boldsymbol{\sigma}) \\ \Rightarrow (\mathbf{r} \cdot \boldsymbol{\sigma}) (\mathbf{n} \cdot \boldsymbol{\sigma}) - (\mathbf{n} \cdot \boldsymbol{\sigma}) (\mathbf{r} \cdot \boldsymbol{\sigma}) &= 0 \\ \Rightarrow \mathbf{r} \times \mathbf{n} &= 0. \end{aligned} \quad (7)$$

which shows that there exist real constants c_1 and c_2 such that $c_1 \mathbf{n} = c_2 \mathbf{r}$.

3 Non-classicalities based on the perturbation under local unitary operations

3.1 Non-classicalities via perturbation

At first, we would like to suppose a local unitary operator U_A is applied to the subsystem A of the state ρ_{AB} , giving the final state:

$$\varrho_{AB} = (U_A \otimes I_2) \rho_{AB} (U_A^\dagger \otimes I_2). \quad (8)$$

Generally, $\varrho_{AB} \neq \rho_{AB}$. So we define

$$D(\rho_{AB}, U_A) := \|\rho_{AB} - \varrho_{AB}\|^2, \quad (9)$$

where $\|X\| = \sqrt{\text{Tr} X X^\dagger}$ denotes the Frobenius norm of the matrix X . $D(\rho_{AB}, U_A)$ is obviously the distance between

the two states before and after the local unitary operation. Considering the extremisation of the distance $D(\rho_{AB}, U_A)$ by the optimization of different unitary operator sets, one can define

$$D_{S_k}(\rho_{AB}) := \max_{U_A \in S_k} D(\rho_{AB}, U_A). \quad (10)$$

which means the maximal distance between the original and the transformed states induced by the local unitary operators belonging to the corresponding unitary operator set S_k with $k = A, T, C$ or S , and

$$\tilde{D}_{S_k}(\rho_{AB}) := \min_{U_A \in S_k} D(\rho_{AB}, U_A). \quad (11)$$

This is the minimal distance that opposes $D_{S_k}(\rho_{AB})$. With these definitions, we will arrive at the following theorem.

Theorem.1. For any a two-qubit state ρ_{AB} ,

$$D_{S_A}(\rho_{AB}) = D_{S_T}(\rho_{AB}) = \lambda_1 + \lambda_2, \quad (12)$$

$$\tilde{D}_{S_A}(\rho_{AB}) = \tilde{D}_{S_C}(\rho_{AB}) = 0, \quad (13)$$

$$D_{S_C}(\rho_{AB}) = D_{MIN}(\rho_{AB}), \quad (14)$$

$$\tilde{D}_{S_T}(\rho_{AB}) = D_{GD}(\rho_{AB}), \quad (15)$$

$$D_{GD}(\rho_{AB}) = \text{Tr} A - \lambda_1, \quad (16)$$

$$D_{MIN}(\rho_{AB}) = \begin{cases} \text{Tr} T T^T - \frac{1}{\|\mathbf{r}\|^2} \mathbf{r}^T T T^T \mathbf{r}, & \mathbf{r} \neq 0 \\ \text{Tr} T T^T - \lambda_3, & \mathbf{r} = 0 \end{cases}, \quad (17)$$

where

$$A = \mathbf{r} \mathbf{r}^T + T T^T. \quad (18)$$

and λ_i is the eigenvalues of A in decreasing order.

Proof. $D(\rho_{AB}, U_A)$ can be explicitly given by

$$D(\rho_{AB}, U_A) = \|\rho_{AB} - \varrho_{AB}\|^2 = 2(\text{Tr} \rho_{AB}^2 - \text{Tr} \rho_{AB} \varrho_{AB}). \quad (19)$$

Some simple calculations reveal that

$$\text{Tr} \rho_{AB}^2 = \frac{1}{4}(2 + \|\mathbf{r}\|^2 + \|\mathbf{s}\|^2 + \sum_{i,j} T_{ij}^2), \quad (20)$$

$$\text{Tr} \rho_{AB} \varrho_{AB} = \frac{1}{4}(2 + \|\mathbf{s}\|^2 + \text{Tr} A + 2\mathbf{n} A \mathbf{n}^T - 2\text{Tr} A \|\mathbf{n}\|^2). \quad (21)$$

By substituting Eqs. (20,21) into Eq. (19), one will obtain

$$\begin{aligned} D(\rho_{AB}, U_A) &= 2\left[\frac{1}{4}(\|\mathbf{r}\|^2 + \sum_{i,j} T_{ij}^2) - \frac{1}{4}(\text{Tr} A + 2\mathbf{n} A \mathbf{n}^T - 2\text{Tr} A \|\mathbf{n}\|^2)\right] \\ &= \mathbf{n} (\text{Tr} A I_3 - A) \mathbf{n}^T. \end{aligned} \quad (22)$$

with A given by Eq. (18).

(1) If $U_A \in S_A$, one will find that the calculation of $\tilde{D}_{S_A}(\rho_{AB})$ is trivial, because one can choose $U_A = I_2$ such that $\tilde{D}_{S_A}(\rho_{AB}) = 0$. In addition, Eq. (22) can be rewritten as

$$\begin{aligned} D(\rho_{AB}, U_A) &= \|\mathbf{n}\|^2 \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} (\text{Tr} A I_3 - A) \frac{\mathbf{n}^T}{\|\mathbf{n}\|} \\ &\leq \lambda_{\max} \|\mathbf{n}\|^2 \end{aligned} \quad (23)$$

$$\leq \lambda_{\max} = \lambda_1 + \lambda_2, \quad (24)$$

where λ_{\max} denotes the sum of the two maximal eigenvalue of A . In addition, the equality in Eq. (23) holds when $\frac{\mathbf{n}}{\|\mathbf{n}\|}$ is chosen as the eigenvector of A corresponding to its maximal eigenvalue and the equality in Eq. (24) is satisfied if we let $n_0 = 0$.

(2) If $U_A \in S_T$, we will obtain $n_0 = 0$ and $\sum_{k=1}^3 n_k^2 = 1$. Thus the bounds of $D(\rho, U_A)$ can be given as follows

$$\begin{aligned} \lambda_2 + \lambda_3 &= \lambda_{\min} \\ &\leq \mathbf{n} (\text{Tr} A I_3 - A) \mathbf{n}^T \\ &\leq \lambda_{\max} = \lambda_1 + \lambda_2, \end{aligned} \quad (25)$$

where λ_{\min} and λ_{\max} are the sum of the two minimal and the sum of the two maximal eigenvalues, respectively, and the equality in Eq. (25) holds if \mathbf{n} is the corresponding eigenvector. Thus we have $D_{S_T}(\rho_{AB}) = \lambda_1 + \lambda_2$ and $\tilde{D}_{S_T}(\rho_{AB}) = \lambda_2 + \lambda_3$.

(3) If $U_A \in S_C$, we have $c_1 \mathbf{n} = c_2 \mathbf{r}$, which implies three cases: a) $\mathbf{n} = 0$; b) $\mathbf{r} = 0$; c) $\mathbf{n} = c\mathbf{r}$, $c \neq 0$. In this case, we can easily find that $\tilde{D}_{S_C}(\rho_{AB}) = 0$ if we choose $\mathbf{n} = 0$. However, if we calculate $D_{S_C}(\rho_{AB})$, we have to consider whether or not $\mathbf{r} = 0$. If b) is satisfied, we will have

$$\begin{aligned} &\mathbf{n} (\text{Tr} A I_3 - A) \mathbf{n}^T \\ &= \text{Tr} T T^T \|\mathbf{n}\|^2 - \mathbf{n} T T^T \mathbf{n}^T \\ &\leq \text{Tr} T T^T - \tilde{\lambda}_3, \end{aligned} \quad (26)$$

with $\tilde{\lambda}_3$ the minimal eigenvalue of $T T^T$. The equality in Eq. (26) is satisfied when \mathbf{n} is the eigenvector of $T T^T$ corresponding to $\tilde{\lambda}_3$. If $\mathbf{r} \neq 0$, c) must be satisfied, so we will arrive at

$$\begin{aligned} &\mathbf{n} (\text{Tr} A I_3 - A) \mathbf{n}^T \\ &= \|\mathbf{n}\|^2 \left(\text{Tr} A - \frac{\mathbf{n}}{\|\mathbf{n}\|} A \frac{\mathbf{n}^T}{\|\mathbf{n}\|} \right) \\ &= c^2 \|\mathbf{r}\|^2 \left(\text{Tr} A - \frac{\mathbf{r}}{\|\mathbf{r}\|} A \frac{\mathbf{r}^T}{\|\mathbf{r}\|} \right) \\ &\leq \text{Tr} A - \frac{\mathbf{r}}{\|\mathbf{r}\|} A \frac{\mathbf{r}^T}{\|\mathbf{r}\|}. \end{aligned} \quad (27)$$

The inequality in Eq. (27) comes from $\|\mathbf{n}\|^2 \leq 1$. Eq. (26) and Eq. (27) are just the MIN introduced in Ref. [5]. The proof is completed.

3.2 Generalized measurement-induced nonlocality

From the above theorem, we find that the optimization on the traceless unitary transformations or the all unitary matrices can lead to a new quantity which can be rewritten as

$$D_{GMIN}(\rho_{AB}) = \lambda_1 + \lambda_2. \quad (28)$$

where λ_i is the eigenvalues of $A = \mathbf{r} \mathbf{r}^T + T T^T$ in decreasing order. Compared with MIN, we would like to call it the generalized measurement-induced nonlocality (GMIN). In order to give an explicit understanding, we would like to introduce several fundamental properties here.

Corollary. 1. For a pure two-qubit state,

$$D_{GMIN}(|\psi\rangle_{AB}) = 2. \quad (29)$$

Proof. Since, for any bipartite pure state $|\psi\rangle_{AB}$, $|\psi\rangle_{AB}$ can be given in a Schmidt decomposition as

$$|\psi\rangle_{AB} = \sigma_1 |00\rangle + \sigma_2 |11\rangle, \quad (30)$$

where $\sigma_1^2 + \sigma_2^2 = 1$, one can obtain the Bloch vector $\mathbf{s} = (0, 0, \sigma_1^2 - \sigma_2^2)^T$ and the correlation tensor

$$T = \begin{pmatrix} 2\sigma_1\sigma_2 & 0 & 0 \\ 0 & -2\sigma_1\sigma_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (31)$$

so we have

$$A = \mathbf{s}\mathbf{s}^T + TT^T = \begin{pmatrix} 4\sigma_1^2\sigma_2^2 & 0 & 0 \\ 0 & 4\sigma_1^2\sigma_2^2 & 0 \\ 0 & 0 & 1 + (\sigma_1^2 - \sigma_2^2)^2 \end{pmatrix}. \quad (32)$$

Therefore, the sum of the two maximal eigenvalues is 2.

Corollary. 2. For any bipartite product state $\rho = \rho_1 \otimes \rho_2$, $D_{GMIN}(\rho) = (4\text{Tr}\rho_1^2 - 2)(4\text{Tr}\rho_2^2 - 1)$. If ρ_1 is an identity, $D_{GMIN}(\rho) = 0$.

Proof. The generic single-qubit can be written as

$$\rho_1 = \frac{1}{2}(1 + \mathbf{x} \cdot \sigma), \quad (33)$$

$$\rho_2 = \frac{1}{2}(1 + \mathbf{y} \cdot \sigma). \quad (34)$$

with Bloch vectors $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$. From here, we will have the bipartite product state

$$\rho = \frac{1}{4}[I_4 + (\mathbf{x} \cdot \sigma) \otimes I_2 + I_2 \otimes (\mathbf{y} \cdot \sigma) + \sum_{i,j} x_i y_j \sigma_i \otimes \sigma_j], \quad (35)$$

Through Theorem 1, the matrix A will be given by $\mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T$. Therefore, the sum of the two maximal eigenvalues is $\mathbf{x}^T \mathbf{x} (1 + \mathbf{y}^T \mathbf{y})$. Using Eqs. (33,34), one can obtain

$$\text{Tr}\rho_1^2 = \frac{1}{4}(2 + \mathbf{x}^T \mathbf{x}), \quad (36)$$

$$\text{Tr}\rho_2^2 = \frac{1}{4}(2 + \mathbf{y}^T \mathbf{y}). \quad (37)$$

It is straightforward $D_{GMIN}(\rho) = (4\text{Tr}\rho_1^2 - 2)(4\text{Tr}\rho_2^2 - 1)$. If ρ_1 is an identity (i.e. $\mathbf{x} = 0$), a simple calculation reveals that $D_{GMIN}(\rho) = 0$.

3.3 Relations between various non-classicalities

From the results given in Theorem 1, we can see that if the optimized local unitary operations are in the set of traceless unitary transformations, the minimum of $D(\rho)$ coincides with the GD. In this sense, it is natural to imagine that the GD is the result of the perturbation of the traceless unitary transformations. Since the MIN is a dual

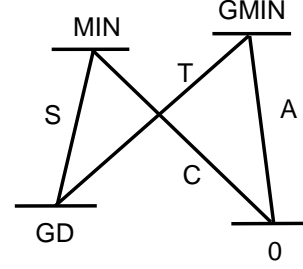


Fig. 1. The relationship between GD, MIN and GMIN. A stands for the set of all unitary operators, C stands for the cyclic unitary operator set, T stands for the traceless unitary operator set and S stands for the special unitary operator set. The horizontal lines denote the values of GD, MIN and GMIN, respectively, and the solid lines connecting them mean the values they connect can be attained by the perturbation with the corresponding unitary operations.

definition of quantum discord, it seems that MIN should also be one result of the perturbation of the traceless unitary transformations. However, based on our theorem, this is not the case. One can find that the dual definition of the GD in the framework of perturbation of local unitary operations is a new quantity, GMIN. From a different perspective, in our theorem, one can find that the MIN is the result of the perturbation of the cyclic unitary transformations. Due to the duality of the GMIN and GD, an intuitive conclusion is that GD should also be the result of the perturbation of cyclic unitary transformations. However, our theorem tells us that the minimum induced by the perturbation of the cyclic unitary transformation is zero instead of the GD. Based on the above analysis, there are two immediate questions. One is what the relation between GMIN and 0 is, and the other is what the relation between MIN and GD is in the framework of perturbed unitary matrices. Even though it seems that GMIN and 0 belong to different kinds of perturbations, they can be unified if we consider the perturbation of all possible unitary transformations. But the relation between MIN and GD seems to be rather complex. From Fig. 2, one can explicitly see that the optimal points corresponding to MIN and GD are both on the same sphere of traceless unitary transformations, and one might naturally ask whether there exists a simple set such as a circle on the sphere connecting the two optimal points such that the two points are still optimal in the sense of the optimization on the circle. Unfortunately, our following theorem shows us that, generally, such a circle does not exist. Of course, at any rate, we can always find such a set despite the complexity. These relations are depicted in Fig. 1.

Theorem. 2. On the sphere of traceless unitary transformations for a general two-qubit state ρ_{AB} there is no

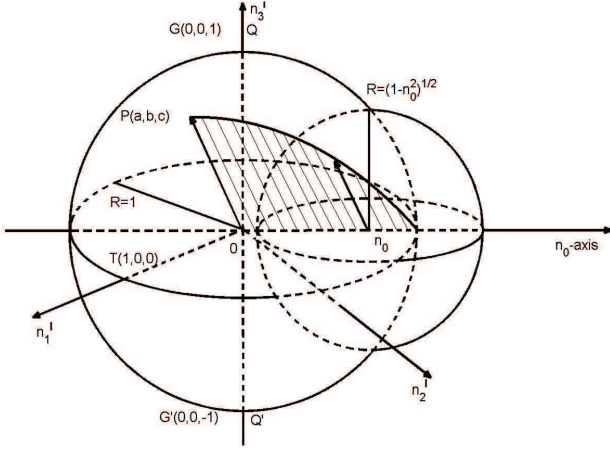


Fig. 2. The set of unitary operations. The sphere with radius $R = 1$ is the traceless operation set, the small sphere with radius $\sqrt{1 - n_0^2}$ is the set of unitary operations corresponding to a given n_0 . The solid line connecting the point $P = (a, b, c)$ and the point $n_0 = 1$ denotes the cyclic set for $|r_A\rangle \neq 0$.

circle whose dual optimization can reach both MIN and GD.

Proof. Based on the proof of Theorem 1, one can find that the optimization problem is given by Eq. (22). Now let $A = U\Lambda U^\dagger$ be the eigenvalue decomposition of A and σ_1, σ_2 and σ_3 be the eigenvalues in descending order, then Eq. (22) will become

$$D(\rho_{AB}, U_A) = \text{Tr} A - (n_1'^2 \sigma_1 + n_2'^2 \sigma_2 + n_3'^2 \sigma_3), \quad (38)$$

where $|n'\rangle = U|n\rangle$, $|r'\rangle = U \frac{|r\rangle}{\|r\|} = (a, b, c)^T$ with $a^2 + b^2 + c^2 = 1$. Thus GD is reached at the point $(\pm 1, 0, 0)$ and MIN is reached at (a, b, c) . In order to find a pathway to realize the relationship between MIN and GD, we can use the points (a, b, c) , $(1, 0, 0)$ (take $(1, 0, 0)$ for analysis, $(-1, 0, 0)$ will give similar results) and a third point (a', b', c') on the sphere to construct a circle to connect MIN to GD. Here we consider the general case, so it is implied that the three points are different from each other. Based on these three points, one can write the equation of the circle as

$$\begin{cases} n_1' + Mn_2' + Nn_3' - 1 = 0 \\ n_1'^2 + n_2'^2 + n_3'^2 = 1 \end{cases}, \quad (39)$$

where

$$M = \frac{c' - c'a + ca' - c}{c'b - cb'}, N = \frac{b' - b'a - b + a'b}{-c'b + cb'}. \quad (40)$$

Let λ and μ be the Lagrangian multiplier, so the Lagrangian equation can be written as

$$\begin{aligned} L = & \text{Tr} A - (n_1'^2 \sigma_1 + n_2'^2 \sigma_2 + n_3'^2 \sigma_3) \\ & + \lambda(n_1' + Mn_2' + Nn_3' - 1) \\ & + \mu(n_1'^2 + n_2'^2 + n_3'^2 - 1). \end{aligned} \quad (41)$$

Taking partial derivative on both sides, one arrives at the following equations:

$$\frac{\partial L}{\partial n_1'} = -2n_1' \sigma_1 + 2\mu n_1' + \lambda, \quad (42)$$

$$\frac{\partial L}{\partial n_2'} = -2n_2' \sigma_2 + 2\mu n_2' + \lambda M, \quad (43)$$

$$\frac{\partial L}{\partial n_3'} = -2n_3' \sigma_3 + 2\mu n_3' + \lambda N, \quad (44)$$

$$\frac{\partial L}{\partial \lambda} = n_1' + Mn_2' + Nn_3' - 1, \quad (45)$$

$$\frac{\partial L}{\partial \mu} = n_1'^2 + n_2'^2 + n_3'^2 - 1. \quad (46)$$

Since both (a, b, c) and $(1, 0, 0)$ are the extreme points, they should satisfy the above five questions. Inserting (a, b, c) into these equations, we arrive at

$$\mu = \frac{-a\sigma_1 + a^2\sigma_1 + b^2\sigma_2 + c^2\sigma_3}{1 - a}, \quad (47)$$

$$\lambda = \frac{2(ab^2\sigma_1 + ac^2\sigma_1 - ab^2\sigma_2 - ac^2\sigma_3)}{1 - a}, \quad (48)$$

$$N = \frac{c[-(-1 + a)a\sigma_1 - b^2\sigma_2 + (-a + a^2 + b^2)\sigma_3]}{a[(b^2 + c^2)\sigma_1 - b^2\sigma_2 - c^2\sigma_3]}, \quad (49)$$

$$M = \frac{b[-(-1 + a)a\sigma_1 - c^2\sigma_3 + (-a + a^2 + b^2)\sigma_2]}{a[(b^2 + c^2)\sigma_1 - b^2\sigma_2 - c^2\sigma_3]}. \quad (50)$$

By considering Eq. (49), Eq. (50) and $a'^2 + b'^2 + c'^2 - 1 = 0$, one finds that (a', b', c') is either (a, b, c) or $(1, 0, 0)$ in the general case, which contradicts our previous requirement that (a', b', c') should be different from (a, b, c) and $(1, 0, 0)$. The proof is completed.

Since we have shown that there is no such set of unitary operations that forms a circle on the sphere of traceless unitary operations, the obvious question is: what set of unitary operations does correspond to the dual optimization of MIN and GD. This is given by our next theorem.

Theorem. 3. The set S_S of the unitary operations corresponding to the dual optimization of MIN and GD is given by

$$S_S = \left\{ U \mid \text{Tr} \left| [\rho, \tilde{U}_C] \right|^2 \geq \text{Tr} |\rho, U|^2 \right\}, \quad (51)$$

where \tilde{U}_C is the optimal unitary matrix leading to MIN.

Proof. Let D denote the distance between ρ and $\rho_f = (U_A \otimes I_2)\rho_{AB}(U_A^\dagger \otimes I_2)$, so

$$\begin{aligned} D_{MIN} - D &= (2\text{Tr}\rho^2 - 2\text{Tr}\rho\tilde{\rho}_f) - (2\text{Tr}\rho^2 - 2\text{Tr}\rho\rho_f) \\ &= 2\text{Tr}\rho\rho_f - 2\text{Tr}\rho\tilde{\rho}_f \\ &= 2\text{Tr}\rho U \rho U^\dagger - 2\text{Tr}\rho \tilde{U}_C \rho \tilde{U}_C^\dagger. \end{aligned} \quad (52)$$

Using

$$\text{Tr} |\rho, U|^2 = 2\text{Tr}\rho^2 - 2\text{Tr}\rho U \rho U^\dagger, \quad (53)$$

Eq. (52) can be rewritten as

$$D_{MIN} - D = \text{Tr} \left| [\rho, \tilde{U}_C] \right|^2 - \text{Tr} |\rho, U|^2. \quad (54)$$

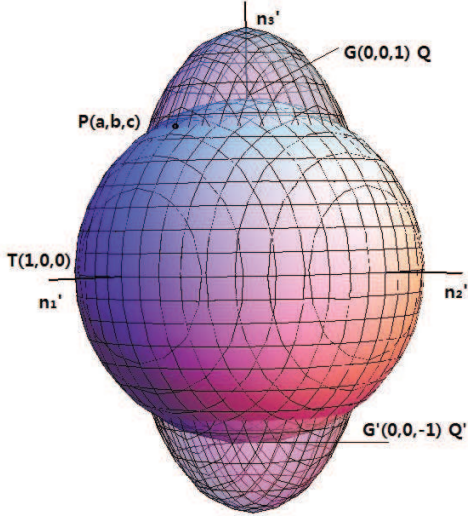


Fig. 3. The special set of unitary operations. The points between the two curves on the sphere where the spheroid and the sphere intersect correspond to the special set of unitary operations that connect MIN and GD. The points P, T, Q, Q', G, G' are analyzed in the text.

Since D_{MIN} is the maximum in the set,

$$\text{Tr} \left[\left[\rho, \tilde{U}_C \right] \right]^2 \geq \text{Tr} [\rho, U]^2. \quad (55)$$

This completes the proof.

In order to provide an intuitive illustration, we would like to sketch the different sets of unitary transformations in Fig. 2. Since any unitary operators on a qubit can be written as Eq. (1), one will find that, given an n_0 , \mathbf{n} characterizes a sphere with radius $\sqrt{1 - n_0^2}$. If we let the horizontal axis denote n_0 in Fig. 2, with n_0 changing from 0 to 1, the sphere with radius 1 corresponding to the traceless unitary operators set S_T will reduce to a point at $n_0 = 1$ corresponding to a unit operator. Denoting the point (a, b, c) by P on the sphere which corresponds to the optimal unitary matrix that attains MIN with $|r_A\rangle \neq 0$. Hence with the change of n_0 , P will undergo a trajectory which is also plotted in the Fig. 2 and denotes the cyclic unitary operator set S_C with $|r_A\rangle \neq 0$. If $|r_A\rangle = 0$, S_C consistent with S_A corresponds to all the spheres with different n_0 included. Based on the above calculation, $(0, 0, \pm 1)$, denoted by $Q(Q')$, is the optimal point corresponding to MIN with $|r_A\rangle = 0$ which is consistent with the optimal point $G(G')$ corresponding to GMIN. In this case, the set $S_S = S_T$. Substituting (a, b, c) and an arbitrary point (a', b', c') into Eq. (52), one will arrive at

$$\frac{a'^2}{\frac{\Delta}{\sigma_1}} + \frac{b'^2}{\frac{\Delta}{\sigma_2}} + \frac{c'^2}{\frac{\Delta}{\sigma_3}} \geq 1, \quad (56)$$

where $\Delta = a^2\sigma_1 + b^2\sigma_2 + c^2\sigma_3$. Eq. (56) describes a spheroid and its outer part which is also shown in Fig. 3. The two curves where the spheroid and the sphere intersect show the potential optimal points for MIN (generally for $|r_A\rangle \neq 0$). The points between the two curves on the

sphere comprise the set S_S with $|r_A\rangle \neq 0$. It is obvious that if the point $P(|r_A\rangle)$ serves as the point of intersection of the curves and the $n'_2 - O - n'_3$ plane, there can exist a great circle that connects MIN and GD. If the point $P(|r_A\rangle)$ serves as the point of intersection of the curves and the $n'_1 - O - n'_3$ plane, there can exist a small circle that connects MIN and GD. Thus it is also apparent that, in the general case, there is no circle on the sphere that directly relates MIN to GD.

4 The non-classicalities of $2 \otimes d$ dimensional quantum systems.

In this section we will discuss the non-classicalities of the qubit-qudit quantum state. Any $2 \otimes d$ dimensional quantum system can be written in the following form [51]

$$\begin{aligned} \rho_{AB} = & \frac{1}{2d} [\mathbf{I} + (\mathbf{r} \cdot \sigma_A) \otimes \mathbf{I} + \sqrt{\frac{d(d-1)}{2}} \mathbf{I} \otimes (\mathbf{s} \cdot \sigma_B) \\ & + \sqrt{\frac{d(d-1)}{2}} \sum_{i,j} T_{ij} \sigma_i \otimes \sigma_j]. \end{aligned} \quad (57)$$

where σ_A and σ_B are the generators of $SU(2)$ and $SU(d)$, \mathbf{r}, \mathbf{s} and T_{ij} are components of the local Bloch vectors and the correlation tensor, respectively. The final state after the local unitary perturbation is

$$\begin{aligned} \varrho_{AB} = & \frac{1}{2d} [\mathbf{I} + (\mathbf{r} \cdot U_A \sigma_A U_A^\dagger) \otimes \mathbf{I} + \sqrt{\frac{d(d-1)}{2}} \mathbf{I} \otimes (\mathbf{s} \cdot \sigma_B) \\ & + \sqrt{\frac{d(d-1)}{2}} \sum_{i,j} T_{ij} U_A \sigma_i U_A^\dagger \otimes \sigma_j]. \end{aligned} \quad (58)$$

Thus we arrive at the following results:

Theorem. 4. For any $2 \otimes d$ dimensional quantum systems ρ_{AB} ,

$$D_{S_A}(\rho_{AB}) = D_{S_T}(\rho_{AB}) = \frac{4}{d^2}(\lambda_1 + \lambda_2), \quad (59)$$

$$\tilde{D}_{S_A}(\rho_{AB}) = \tilde{D}_{S_C}(\rho_{AB}) = 0, \quad (60)$$

$$D_{S_C}(\rho_{AB}) = \frac{2(d-1)}{d} D_{MIN}(\rho_{AB}), \quad (61)$$

$$\tilde{D}_{S_T}(\rho_{AB}) = \frac{4}{d^2} D_{GD}(\rho_{AB}), \quad (62)$$

$$D_{GD}(\rho_{AB}) = \text{Tr} A - \lambda_1, \quad (63)$$

$$D_{MIN}(\rho_{AB}) = \begin{cases} \text{Tr} T T^T - \frac{1}{\|\mathbf{r}\|^2} \mathbf{r}^T T T^T \mathbf{r}, & \mathbf{r} \neq 0 \\ \text{Tr} T T^T - \lambda_3, & \mathbf{r} = 0 \end{cases}. \quad (64)$$

where $A = \mathbf{r} \mathbf{r}^T + \frac{d(d-1)}{2} T T^T$ and λ_i is the eigenvalues of A in decreasing order.

Proof. $D(\rho_{AB}, U_A)$ can be explicitly given by

$$D(\rho_{AB}, U_A) = \|\rho_{AB} - \varrho_{AB}\|^2 = 2(\text{Tr} \rho_{AB}^2 - \text{Tr} \rho_{AB} \varrho_{AB}). \quad (65)$$

After the simple calculations, one has

$$\begin{aligned} \text{Tr}\rho_{AB}^2 &= \frac{1}{4d^2} \text{Tr}[\mathbf{I} + \mathbf{r}\mathbf{r}^T \sigma_A \sigma_A \otimes \mathbf{I} \\ &\quad + \frac{d(d-1)}{2} \sum_{i,j} \sum_{m,n} T_{ij} T_{mn} \sigma_i \sigma_m \otimes \sigma_j \sigma_n] \\ &\quad + \frac{d(d-1)}{2} \mathbf{I} \otimes \mathbf{S}\mathbf{S}^T \sigma_B \sigma_B, \end{aligned} \quad (66)$$

$$\begin{aligned} \text{Tr}\rho_{AB} \varrho_{AB} &= \frac{1}{4d^2} \text{Tr}[\mathbf{I} + \mathbf{r}\mathbf{r}^T \sigma_A U_A \sigma_A U_A^\dagger \otimes \mathbf{I} \\ &\quad + \frac{d(d-1)}{2} \sum_{i,j} \sum_{m,n} T_{ij} T_{mn} \sigma_i U_A \sigma_m U_A^\dagger \otimes \sigma_j \sigma_n] \\ &\quad + \frac{d(d-1)}{2} \mathbf{I} \otimes \mathbf{S}\mathbf{S}^T \sigma_B \sigma_B. \end{aligned} \quad (67)$$

Substituting Eqs. (66,67) into Eq. (65), one obtains

$$D(\rho_{AB}, U_A) = \frac{4}{d^2} \mathbf{n}(\text{Tr}AI - A)\mathbf{n}^T \quad (68)$$

where A is given by $A = \mathbf{r}\mathbf{r}^T + \frac{d(d-1)}{2} TT^T$.

(1) It is trivial to show that $\tilde{D}_{S_A}(\rho_{AB}) = 0$ when $U_A \in S_A$. In addition, Eq. (68) can be rewritten as

$$\begin{aligned} D(\rho_{AB}, U_A) &= \frac{4}{d^2} \|\mathbf{n}\|^2 \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} (\text{Tr}AI - A) \frac{\mathbf{n}^T}{\|\mathbf{n}\|} \\ &\leq \frac{4}{d^2} \lambda_{\max} \|\mathbf{n}\|^2 \\ &\leq \frac{4}{d^2} \lambda_{\max} = \frac{4}{d^2} (\lambda_1 + \lambda_2), \end{aligned} \quad (69)$$

where λ_{\max} denotes the sum of the two maximal eigenvalues of A . In addition, the equality in Eq. (69) holds if and only if $\frac{\mathbf{n}}{\|\mathbf{n}\|}$ is chosen as the eigenvector of A corresponding to its maximal eigenvalue. And if we let $n_0 = 0$, the equality in Eq. (70) will be reached.

(2) When $U_A \in S_T$, it means that $n_0 = 0$ and $\sum_{k=1}^3 n_k^2 = 1$. Thus the upper and lower bounds on $D(\rho_{AB}, U_A)$ will be given as follows:

$$\begin{aligned} \lambda_2 + \lambda_3 &= \lambda_{\min} \\ &\leq \frac{4}{d^2} \mathbf{n}(\text{Tr}AI - A)\mathbf{n}^T \\ &\leq \frac{4}{d^2} \lambda_{\max} = \frac{4}{d^2} (\lambda_1 + \lambda_2), \end{aligned} \quad (71)$$

where λ_{\min} and λ_{\max} are the sum of the two minimal and the sum of the two maximal eigenvalues of A , respectively. If \mathbf{n} takes the corresponding eigenvector, the equality in Eq. (71) holds. In this case, $D_{S_T}(\rho_{AB}) = \frac{4}{d^2} (\lambda_1 + \lambda_2)$ and $\tilde{D}_{S_T}(\rho_{AB}) = \frac{4}{d^2} (\lambda_2 + \lambda_3)$. This result coincides with the quantum discord for a qubit-qudit system[34].

(3) When $U_A \in S_C$, we have from Lemma 1 that $c_1 \mathbf{n} = c_2 \mathbf{r}$, which also implies three cases: a) $\mathbf{n} = 0$; b) $\mathbf{r} = 0$; c) $\mathbf{n} = c\mathbf{r}$, $c \neq 0$, just like in the two-qubit quantum states. In this case, if we choose $\mathbf{n} = 0$ we can easily find that $\tilde{D}_{S_C}(\rho_{AB}) = 0$. However, if we want to calculate

$D_{S_C}(\rho_{AB})$, we have to consider whether $\mathbf{r} = 0$ or not. If b) is satisfied, we will have

$$\begin{aligned} &\frac{4}{d^2} \mathbf{n}(\text{Tr}AI - A)\mathbf{n}^T \\ &= \frac{2(d-1)}{d} (\text{Tr}TT^T \|\mathbf{n}\|^2 - \mathbf{n}TT^T\mathbf{n}^T) \\ &\leq \frac{2(d-1)}{d} (\text{Tr}TT^T - \tilde{\lambda}_3), \end{aligned} \quad (72)$$

with $\tilde{\lambda}_3$ the minimal eigenvalue of TT^T . The equality in Eq. (72) is satisfied when \mathbf{n} is the eigenvector of TT^T corresponding to $\tilde{\lambda}_3$. If $\mathbf{r} \neq 0$, c) has to be satisfied. Hence, we arrive at

$$\begin{aligned} &\frac{4}{d^2} \mathbf{n}(\text{Tr}AI - A)\mathbf{n}^T \\ &= \frac{4}{d^2} \|\mathbf{n}\|^2 \left(\text{Tr}A - \frac{\mathbf{n}}{\|\mathbf{n}\|} A \frac{\mathbf{n}^T}{\|\mathbf{n}\|} \right) \\ &= \frac{4}{d^2} c^2 \|\mathbf{r}\|^2 \left(\text{Tr}A - \frac{\mathbf{r}}{\|\mathbf{r}\|} A \frac{\mathbf{r}^T}{\|\mathbf{r}\|} \right) \\ &\leq \frac{2(d-1)}{d} \left(\text{Tr}TT^T - \frac{\mathbf{r}}{\|\mathbf{r}\|} TT^T \frac{\mathbf{r}^T}{\|\mathbf{r}\|} \right). \end{aligned} \quad (73)$$

The inequality in Eq. (73) comes from $\|\mathbf{n}\|^2 \leq 1$. Eq. (72) and Eq. (73) are just the MIN introduced in Ref. [5].

5 Discussions and Conclusion

Before concluding, we would like to make a brief comparison between the results of Ref. [48] or Ref. [49] and our results. Ref. [48] and Ref. [49] showed that the geometric measure of quantum correlations can be defined not only by taking the distance from the state being considered and its image under a generic local unitary operation, but also by minimizing this distance over all traceless local unitary operations. However, in our work we employ the same method, i.e., the perturbation of local unitary operation, not only to obtain geometric discord as those in Ref. [48] and Ref. [49], but also to obtain the dual definitions of discord, i.e., the MIN and GMIN. In particular, we mainly emphasize that for a given quantum correlation, its dual definition of quantum correlation is not unique based on the local unitary perturbations. This has been explicitly illustrated in Fig. 1. Regarding local unitary operations, we have considered various types of unitary operation sets such as S_A , S_C , S_T and S_S , while others only considered a few of them.

To sum up, we have studied non-classicalities based on the perturbation of local unitary operations. We find that both GD and MIN can be understood in this way. However, even though GD and MIN are the two dual definitions in the framework of their original definitions, they cannot be connected by a simple set of unitary operations in the sense of the perturbation of local unitary operations. On the contrary, it is shown that they are connected in a strange way by the set S_S . In addition, we find that

all the non-classicalities described in the paper have no unique explanation based on the perturbation of the local unitary operations, which naturally leads to their dual definitions being quite different, that is, the dual definition is strongly dependent on the perturbing unitary operation set. We hope this will shed new light on quantum correlations. Lastly, we would like to say that the traceless local unitary operation is only sufficient for qubit systems and one must consider the completely non-degenerate traceless unitary operations for high dimensional systems. In addition, all the quantities given in this paper are based on the Frobenius Norm, which is not-contractive. Considering the recent exciting results based on skew information [42], we look forward to attempting to relate local unitary perturbation to skew-information based measures.

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